

BETTI TABLES OF p -BOREL-FIXED IDEALS

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ABSTRACT. In this note we provide a counter-example to a conjecture of K. Pardue [Thesis, Brandeis University, 1994.], which asserts that if a monomial ideal is p -Borel-fixed, then its \mathbb{N} -graded Betti table, after passing to any field does not depend on the field. More precisely, we show that, for any monomial ideal I in a polynomial ring S over the ring \mathbb{Z} of integers and for any prime number p , there is a p -Borel-fixed monomial S -ideal J such that a region of the multigraded Betti table of $J(S \otimes_{\mathbb{Z}} \ell)$ is in one-to-one correspondence with the multigraded Betti table of $I(S \otimes_{\mathbb{Z}} \ell)$ for all fields ℓ of arbitrary characteristic. There is no analogous statement for Borel-fixed ideals in characteristic zero. Additionally, the construction also shows that there are p -Borel-fixed ideals without any minimal cellular resolution.

1. INTRODUCTION

Let x_1, \dots, x_n be indeterminates over the ring \mathbb{Z} of integers and $S = \mathbb{Z}[x_1, \dots, x_n]$. Let p be zero or a prime number. For any field \mathbb{k} , the general linear group $\mathrm{GL}_n(\mathbb{k})$ acts on $S \otimes_{\mathbb{Z}} \mathbb{k}$. Say that a monomial S -ideal I is p -Borel-fixed if $I(S \otimes_{\mathbb{Z}} \mathbb{k})$ is fixed under the action of the Borel subgroup of $\mathrm{GL}_n(\mathbb{k})$ consisting of all the upper triangular invertible matrices over \mathbb{k} for any infinite field \mathbb{k} of characteristic p . (This definition does not depend on the choice of \mathbb{k} ; see Proposition 2.6.)

Let I be any monomial S -ideal. In Theorem 3.2 we will show that for any prime number p , there exists a (monomial) S -ideal J that is p -Borel-fixed and that, for any field ℓ , there is a region (independent of ℓ) in the multigraded Betti table of $J(S \otimes_{\mathbb{Z}} \ell)$ (as a module over $S \otimes_{\mathbb{Z}} \ell$) that is determined by the multigraded Betti table of $I(S \otimes_{\mathbb{Z}} \ell)$. This shows that, homologically, the class of Borel-fixed ideals in positive characteristic is as bad as the class of all monomial ideals.

There is a combinatorial characterization of p -Borel-fixed S -ideals; see Proposition 2.6. It follows from this characterization that if I is 0-Borel-fixed, then $I(S \otimes_{\mathbb{Z}} \ell)$ is Borel-fixed for all fields ℓ , irrespective of $\mathrm{char} \ell$; the converse is not true. The Eliahou-Kervaire complex [EK90, Theorem 2.1] gives S -free resolutions of 0-Borel-fixed ideals in S , which specialize to a minimal resolutions over any field ℓ . In particular, the \mathbb{N}^n -graded Betti table (and, hence, the \mathbb{N} -graded Betti table) of a 0-Borel-fixed S -ideal remains unchanged after passing to any field. On the other hand, if we only assume that I is p -Borel-fixed, with $p > 0$, then little is known about minimal resolutions of $I(S \otimes_{\mathbb{Z}} \ell)$ for some field ℓ , including when $\mathrm{char} \ell = p$.

A systematic study of Borel-fixed ideals in positive characteristic was begun by K. Pardue [Par94]. In positive characteristic, Proposition 2.6 was proved by him. He gave a conjectural formula for the (Castelnuovo-Mumford) regularity of principal p -Borel-fixed ideals. A. Aramova and J. Herzog [AH97, Theorem 3.2] showed that the conjectured formula is a lower bound for regularity; Herzog and D. Popescu [HP01, Theorem 2.2] finished the proof of the conjecture by showing that it is also an upper bound. V. Ene, G. Pfister and Popescu [EPP00] determined Betti numbers and Koszul homology of a class of Borel-fixed ideals in $\mathbb{k}[x_1, \dots, x_n]$, where $\mathrm{char} \mathbb{k} = p > 0$, which they called ‘ p -stable’.

Our main result (Theorem 3.2) arose in the following way. It is known that the Eliahou-Kervaire resolution is cellular [Mer10]. Using algebraic discrete Morse theory, M. Jöllenbeck and V. Welker constructed minimal cellular free resolutions of principal Borel-fixed ideals in positive characteristic [JW09, Chapter 6]; see, also, [Sin08]. We were trying to see whether this extends to more general p -Borel-fixed ideals when we realized the possibility of the existence of p -Borel-fixed ideals whose Betti tables might depend on the characteristic.

Key words and phrases. Graded free resolutions, positive characteristic, Borel-fixed ideals, cellular resolutions.

The work of the first author was supported by a grant from the Simons Foundation (209661 to G. C.). In addition, both the authors thank Mathematical Sciences Research Institute, Berkeley CA, where part of this work was done, for support and hospitality during Fall 2012.

As a corollary of our construction and the result of M. Velasco [Vel08] that there are monomial ideals without minimal cellular resolution, we conclude that there are p -Borel-fixed ideals that do not admit minimal cellular resolutions.

We remarked earlier that the \mathbb{N} -graded Betti table of a 0-Borel-fixed S ideal remains identical over any field. Pardue [Par94, Conjecture V.4, p. 43] conjectured that this is true also for p -Borel-fixed ideals; see Conjecture 2.7 for the statement. (This conjecture also appears in [PS08, 4.3].) There has been some evidence that the conjecture is true. If J is a p -Borel-fixed S -ideal, then the projective dimension of $J(S \otimes_{\mathbb{Z}} \ell)$ is determined by the largest i such that x_i divides some minimal monomial generator of J . The regularity of $J(S \otimes_{\mathbb{Z}} \ell)$ does not depend on ℓ [Par94, Corollary VI.9]; this is part of the motivation for Pardue to make this conjecture. Later, Popescu [Pop05] showed that the extremal Betti numbers of $J(S \otimes_{\mathbb{Z}} \ell)$ does not depend on ℓ . However, Example 3.7 shows that the conjecture is not true.

We thank Ezra Miller for helpful comments. The computer algebra system Macaulay2 [M2] provided valuable assistance in studying examples.

2. PRELIMINARIES

We begin with some preliminaries on estimating the graded Betti numbers of monomial ideals and on p -Borel-fixed ideals. By \mathbb{N} we denote the set of non-negative integers. When we say that p is a prime number, we will mean that $p > 0$. By $\mathbf{e}_1, \dots, \mathbf{e}_n$, we mean the standard vectors in \mathbb{N}^n .

Let A be an \mathbb{N}^d -graded polynomial ring (for some integer $d \geq 1$) over a field \mathbb{k} , with $A_{\mathbf{0}} = \mathbb{k}$. Let M be an \mathbb{N}^d -graded A -module. (All the modules that we deal with in this paper are ideals or quotients of ideals.) The \mathbb{N}^d -graded Betti numbers of M are $\beta_{i,\mathbf{a}}^A(M) := \dim_{\mathbb{k}} \operatorname{Tor}_i^A(M, \mathbb{k})_{\mathbf{a}}$. The \mathbb{N}^d -graded Betti table of M is the element $(\beta_{i,\mathbf{a}}^A(M))_{i,\mathbf{a}} \in \mathbb{N} \times \mathbb{N}^d$. For $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{N}^d$, we write $|\mathbf{a}| = a_1 + \dots + a_d$.

Notation 2.1. Let A be a Noetherian ring and z an indeterminate over A . Let $B = A[z]$; it is a graded A -algebra with $\deg z = 1$. For a graded B -ideal I , define A -ideals $I_{\langle i \rangle} = ((I : z^i) \cap A)$, for all $i \in \mathbb{N}$. \square

Note that for all $i \in \mathbb{N}$, $I_{\langle i \rangle} \subseteq I_{\langle i+1 \rangle}$. Moreover, since B is Noetherian, $I_{\langle i \rangle} = I_{\langle i+1 \rangle}$ for all $i \gg 0$.

Lemma 2.2. *Adopt Notation 2.1. Suppose that A is a \mathbb{N}^d -graded polynomial ring (for some integer $d \geq 1$) over a field \mathbb{k} of arbitrary characteristic, with $A_{\mathbf{0}} = \mathbb{k}$. Let I be a graded B -ideal (in the natural \mathbb{N}^{d+1} -grading of B). Then for all $\mathbf{a} \in \mathbb{N}^d$,*

$$\beta_{i,(\mathbf{a},j)}^B(I) = \begin{cases} 0, & \text{if } j < 0, \\ \beta_{i,\mathbf{a}}^A(I_{\langle 0 \rangle}), & \text{if } j = 0, \text{ and} \\ \beta_{i-1,\mathbf{a}}^A(I_{\langle j \rangle} / I_{\langle j-1 \rangle}), & \text{otherwise.} \end{cases}$$

Proof. Fix $\mathbf{a} \in \mathbb{N}^d$. Let $M := I_{\langle 0 \rangle} B \oplus \bigoplus_{l \geq 1} (I_{\langle l \rangle} / I_{\langle l-1 \rangle}) \otimes_A B(-(\mathbf{0}, l))$. We need to prove that $\beta_{i,(\mathbf{a},j)}^B(I) = \beta_{i,(\mathbf{a},j)}^B(M)$ for all i, j . There are two exact sequences

$$\begin{aligned} 0 &\longrightarrow I(-(\mathbf{0}, 1)) \xrightarrow{z} I \longrightarrow I/zI \longrightarrow 0, \\ 0 &\longrightarrow M(-(\mathbf{0}, 1)) \xrightarrow{z} M \longrightarrow I/zI \longrightarrow 0. \end{aligned}$$

The maps $\operatorname{Tor}_i^B(I(-(\mathbf{0}, 1)), \mathbb{k}) \xrightarrow{z} \operatorname{Tor}_i^B(I, \mathbb{k})$ and $\operatorname{Tor}_i^B(M(-(\mathbf{0}, 1)), \mathbb{k}) \xrightarrow{z} \operatorname{Tor}_i^B(M, \mathbb{k})$ are zero. Therefore, for all i and for all $j > 0$.

$$(2.3) \quad \beta_{i,(\mathbf{a},j)}^B(I) + \beta_{i-1,(\mathbf{a},j-1)}^B(I) = \beta_{i,(\mathbf{a},j)}^B(I/zI) = \beta_{i,(\mathbf{a},j)}^B(M) + \beta_{i-1,(\mathbf{a},j-1)}^B(M).$$

Note that outside a bounded rectangle inside \mathbb{Z}^2 , the functions $(i, j) \mapsto \beta_{i,(\mathbf{a},j)}^B(I)$ and $(i, j) \mapsto \beta_{i,(\mathbf{a},j)}^B(M)$ take the value zero. Therefore it follows from (2.3) that $\beta_{i,(\mathbf{a},j)}^B(I) = \beta_{i,(\mathbf{a},j)}^B(M)$ for all i, j . \square

Definition 2.4. Adopt Notation 2.1. Let $d = (d_0 < d_1 < \dots)$ be an increasing sequence of natural numbers. Define an operation Φ_d on graded B -ideals by setting $\Phi_d(I)$ to be the B -ideal generated by $\bigoplus_{i \in \mathbb{N}} I_{\langle i \rangle} z^{d_i}$. \square

Proposition 2.5. *Adopt the hypothesis of Lemma 2.2. Then*

$$\beta_{i,(\mathbf{a},j)}(\Phi_d(I)) = \begin{cases} \beta_{i,(\mathbf{a},l)}(I), & \text{if } j = d_l \\ 0, & \text{otherwise.} \end{cases}$$

Proof. This follows immediately by noting that, for all $j \in \mathbb{N}$, $(\Phi_d(I))_{\langle j \rangle} = I_{\langle l \rangle}$ where l is such that $d_l \leq j < d_{l+1}$. (If $d_0 > 0$, then $(\Phi_d(I))_{\langle j \rangle} = 0$ for all $0 \leq j < d_0$.) \square

Borel-fixed ideals. For the duration of this paragraph and Proposition 2.6, assume that p is zero or a positive prime number. Given two non-negative integers a and b , say that $a \preceq_p b$ if $\binom{b}{a} \not\equiv 0 \pmod{p}$. Then there is the following characterization of Borel-fixed ideals; for positive characteristic, it was proved by Pardue [Par94, Proposition II.4]. For details, see [Eis95, Section 15.9.3].

Proposition 2.6 ([Eis95, Theorem 15.23]). *Let \mathbb{k} be an infinite field of characteristic p . An ideal I of $\mathbb{k}[x_1, \dots, x_n]$ is Borel fixed if and only if I is a monomial ideal and for all $i < j$ and for all monomial minimal generators m of I , $(x_i/x_j)^s m \in I$ for all $s \preceq_p t$ where t is the largest integer such that $x_j^t \mid m$.*

Conjecture 2.7 ([Par94, Conjecture V.4, p. 43]). *Let p be a prime number. Let I be a p -Borel-fixed monomial S -ideal. Then the \mathbb{N} -graded Betti table of $I(S \otimes_{\mathbb{Z}} \ell)$ is independent of $\text{char } \ell$ (equivalently, ℓ) for all fields ℓ (of arbitrary characteristic).*

3. CONSTRUCTION

Recall that $S = \mathbb{Z}[x_1, \dots, x_n]$ and that I is a monomial S -ideal. Fix a prime number p and let \mathbb{k} be any field of characteristic p . We now describe an algorithm that constructs an S -ideal J such that $J(S \otimes_{\mathbb{Z}} \mathbb{k})$ is Borel-fixed.

Construction 3.1. Input: A monomial S -ideal I . Set $i = 1$ and $J_0 = I$.

- (i) Pick r_i an upper bound for $\text{reg}_{(S \otimes_{\mathbb{Z}} \ell)}(J_{i-1}(S \otimes_{\mathbb{Z}} \ell))$ that is independent of ℓ .
 - (ii) Pick a positive integer e_i such that $p^{e_i} > r_i$. Let $d = (0 < p^{e_i} < 2p^{e_i} < 3p^{e_i} < \dots)$. Set $J_i = \Phi_d(J_{i-1} + (x_i^{p^{e_i}}))$ with $A = \mathbb{Z}[x_1, \dots, x_i, x_{i+2}, \dots, x_n]$, $z = x_{i+1}$ and $B = S$ (Definition 2.4). Note that we are adding a large power of x_i but modifying the resulting ideal with respect to x_{i+1} .
 - (iii) If $i = n - 1$ then set $J = J_i$ and exit, else replace i by $i + 1$ and go to Step (i).
- Output: A monomial S -ideal J . \square

Before we state our theorem, we need to identify a region of the \mathbb{N}^n -graded Betti table of $J(S \otimes_{\mathbb{Z}} \ell)$ that captures the \mathbb{N}^n -graded Betti table of $I(S \otimes_{\mathbb{Z}} \ell)$. a map that describes the relation. Let $\mathcal{A} = \{\mathbf{a} : |\mathbf{a}| \leq r_1\}$ (with r_1 as in Step (i)) and $\mathcal{B} = \{\mathbf{b} : b_j < p^{e_j} - 1\}$.

Theorem 3.2. *The ideal J is p -Borel-fixed. Moreover, there is an injective map $\psi : \mathcal{A} \longrightarrow \mathcal{B}$ such that for all fields ℓ (of arbitrary characteristic), for all $1 \leq i \leq n$, and for all $\mathbf{b} \in \mathcal{B}$,*

$$\beta_{i,\mathbf{b}}^{S \otimes_{\mathbb{Z}} \ell}(J(S \otimes_{\mathbb{Z}} \ell)) = \begin{cases} \beta_{i,\psi^{-1}(\mathbf{b})}^{S \otimes_{\mathbb{Z}} \ell}(I(S \otimes_{\mathbb{Z}} \ell)), & \text{if } \mathbf{b} \in \text{Im } \psi, \\ 0, & \text{otherwise.} \end{cases}$$

Let us make some remarks about the construction. In Step (i), we may, for example, take r_i to be the degree of the least common multiple of the minimal monomial generators of J_{i-1} ; that this is a bound for regularity (independent of characteristic) follows from the Taylor resolution. There are stronger bounds, e.g., the largest degree of a minimal generator of the lex-segment ideal with the same Hilbert function as $J_{i-1}(S \otimes_{\mathbb{Z}} \ell)$. The construction, as presented, gives an ideal with (x_1, \dots, x_n) as a minimal prime. To avoid this, one may insert a check at Step (iii) whether $J_i(S \otimes_{\mathbb{Z}} \mathbb{Z}/p)$ is Borel-fixed using Proposition 2.6. The algorithm will, then, terminate before or at the stage $i = m - 1$ where $m = \max\{i : x_i \text{ divides a minimal monomial generator of } I\}$.

We begin with a description of the change in the \mathbb{N}^n -graded Betti table at Step (ii). For the sake of readability, we will abbreviate, for monomial S -ideals \mathbf{a} , $\beta_{i,\mathbf{b}}^{S \otimes_{\mathbb{Z}} \ell}(\mathbf{a}(S \otimes_{\mathbb{Z}} \ell))$ by $\beta_{i,\mathbf{b}}^{\ell}(\mathbf{a})$ and $\text{reg}_{(S \otimes_{\mathbb{Z}} \ell)}(\mathbf{a}(S \otimes_{\mathbb{Z}} \ell))$ by $\text{reg}_{\ell}(\mathbf{a})$, from here till the end of the proof of theorem.

Discussion 3.3. The choice of e_j implies that $(J_{j-1} :_S x_j^{p^{e_j}}) = (J_{j-1} :_S x_j^\infty)$ and that $S/(J_{j-1} + (x_j^{p^{e_j}}))$ is minimally resolved (over any field ℓ) by the mapping cone of the comparison map from the minimal resolution of $S/(J_{j-1} :_S x_j^{p^{e_j}})$ to the minimal resolution of S/J_{j-1} . Moreover, if $|\mathbf{b}| \geq i + p^{e_j}$ then $|\mathbf{b}| > i + \text{reg}_\ell(J_{j-1})$, so the Betti numbers $\beta_{*,\mathbf{b}}^\ell(J_{j-1} + (x_j^{p^{e_j}}))$ are determined by resolution of $(S/(J_{j-1} :_S x_j^\infty))(-p^{e_j}\mathbf{e}_j)$; hence, in particular, for such \mathbf{b} , if $\beta_{*,\mathbf{b}}^\ell(J_{j-1} + (x_j^{p^{e_j}})) \neq 0$, then $b_j \geq p^{e_j}$. Putting this together, we obtain the following:

$$\beta_{i,\mathbf{b}}^\ell(J_{j-1} + (x_j^{p^{e_j}})) = \begin{cases} \beta_{i,\mathbf{b}}^\ell(J_{j-1}), & \text{if } |\mathbf{b}| < i + p^{e_j}, \\ \beta_{i,\mathbf{b}-p^{e_j}\mathbf{e}_j}^\ell(J_{j-1} :_S x_j^\infty), & \text{otherwise.} \end{cases}$$

Proposition 2.5 implies that for all $\mathbf{b} \in \mathbb{N}^n$,

$$(3.4) \quad \beta_{i,\mathbf{b}}^\ell(J_j) = \begin{cases} \beta_{i,\mathbf{b}'}^\ell(J_{j-1}), & \text{if } p^{e_j} \mid b_{j+1} \text{ and } b_j < p^{e_j}, \\ \beta_{i,\mathbf{b}''}^\ell(J_{j-1} :_S x_j^\infty), & \text{if } p^{e_j} \mid b_{j+1} \text{ and } b_j \geq p^{e_j}, \\ 0, & \text{otherwise,} \end{cases}$$

where write $\mathbf{b}' = \mathbf{b} - (b_{j+1} - \frac{b_{j+1}}{p^{e_j}})\mathbf{e}_{j+1}$ and $\mathbf{b}'' = \mathbf{b}' - p^{e_j}\mathbf{e}_j$. We can recover the \mathbb{N}^n -graded Betti table of J_{j-1} from the \mathbb{N}^n -graded Betti table of J_j . To make this precise, suppose that $\beta_{i,\mathbf{b}}^\ell(J_j) \neq 0$. Then the resulting dichotomous situation from (3.4) has the following re-interpretation:

$$(3.5) \quad \begin{aligned} b_j < p^{e_j} & \quad \text{if and only if} \quad \beta_{i,\mathbf{b}}^\ell(J_j) = \beta_{i,\mathbf{b}'}^\ell(J_{j-1}), \\ b_j \geq p^{e_j} & \quad \text{if and only if} \quad \beta_{i,\mathbf{b}}^\ell(J_j) = \beta_{i,\mathbf{b}'}^\ell(J_{j-1} :_S x_j^\infty). \end{aligned}$$

We will not explicitly construct the map ψ , but will observe that it can be done by repeated use of (3.5). \square

Proof of the theorem. Without loss of generality, we may assume that \mathbf{k} is infinite. Let $x_1^{a_1} \cdots x_n^{a_n}$ be a minimal monomial generator of J . For all $1 \leq i \leq n-1$, a_{i+1} is a multiple of p^{e_i} and $x_i^{p^{e_i}} \in J$. Note that for all integers $l \geq 1$, if $m \prec_p lp^{e_i}$ for some integer m , then m is a multiple of p^{e_i} . Therefore the first assertion follows from Proposition 2.6. The assertion about the Betti numbers $\beta_{i,\mathbf{b}}^\ell(J)$ follows from Discussion 3.3, repeatedly applying (3.5). \square

Fix a positive integer r . Instead of choosing r_1 in a way that depends on J_0 in Step (i) of Construction 3.1, we may set $r_1 = r$ and run the algorithm to obtain a variant of Theorem 3.2 that applies uniformly to all monomial S -ideals.

Remark 3.6 (Minimal cellular resolutions). Suppose that $J(S \otimes_{\mathbb{Z}} \ell)$ has a minimal cellular resolution. (See [MS05, Chapter 4] for the definition and properties of cellular resolutions.) Denote by X the CW-complex on which the resolution of $J(S \otimes_{\mathbb{Z}} \ell)$ is supported. Let $\mathbf{b} = (p^{e_1} - 1, p^{e_1} - 1, \dots, p^{e_{n-1}} - 1, b)$ where b is any sufficiently large number. It follows from (3.5) and [MS05, Proposition 4.5] that the sub-complex $X_{\leq \mathbf{b}}$ is acyclic and hence supports a minimal cellular resolution of $I(S \otimes_{\mathbb{Z}} \ell)$. However, there are ideals whose minimal resolution is not cellular [Vel08]. Hence there are p -Borel-fixed S -ideals J such that the minimal resolution of $J(S \otimes_{\mathbb{Z}} \ell)$ is not cellular. However if $I(S \otimes_{\mathbb{Z}} \ell)$ has a minimal cellular resolution, then $J(S \otimes_{\mathbb{Z}} \ell)$ also will have one, for the following reason. At stage j of Construction 3.1, if J_{j-1} has a minimal cellular resolution supported on a CW-complex X , then $(J_{j-1} : (x_j^{p^{e_j}}))$ has a minimal cellular resolution supported on a sub-CW-complex Y of X , since $(J_{j-1} : (x_j^{p^{e_j}})) = (J_{j-1} : (x_j^\infty))$. Then the mapping cone of the inclusion $Y \rightarrow X$ supports the mapping cone of the corresponding free resolutions; we thank Ezra Miller for pointing this out to us. Now, as we observed in Discussion 3.3, this mapping cone is a minimal resolution of $(J_{j-1} + (x_j^{p^{e_j}}))$. On the other hand, the CW-complex that supports the minimal resolution $(J_{j-1} + (x_j^{p^{e_j}}))$ also supports the minimal resolution J_j ; applying Φ_d only amounts to relabelling the underlying CW-complex. \square

Example 3.7 (Counter-examples to Conjecture 2.7). Note that, since graded Betti numbers are upper-semicontinuous functions of characteristic, for an S -ideal J , the \mathbb{N} -graded Betti table of $(J(S \otimes_{\mathbb{Z}} \ell))$ depends on $\text{char } \ell$ if and only if the \mathbb{N}^n -graded Betti table depends on $\text{char } \ell$. Let I be any monomial S -ideal such that

its \mathbb{N}^n -graded Betti table depends on $\text{char } \ell$. Let p be any prime number and \mathbb{k} any field of characteristic p . Let J be the ideal from Construction 3.1. Then $J(S \otimes_{\mathbb{Z}} \ell)$ is Borel-fixed while its \mathbb{N}^n -graded Betti table depends on $\text{char } \ell$. As a specific example, we consider the Reisner triangulation of the real projective plane [BH93, Section 5.3]. We have

$$S = \mathbb{Z}[x_1, \dots, x_6]$$

$$I = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_2x_4x_5, x_3x_4x_5, x_2x_3x_6, x_1x_4x_6, x_3x_4x_6, x_1x_5x_6, x_2x_5x_6).$$

With $e_1 = 3$, $e_2 = 5$, $e_3 = 7$, $e_4 = 9$, and $e_5 = 11$, we obtain

$$J = (x_1^8, x_2^{32}, x_1x_2^8x_3^{32}, x_3^{128}, x_1x_2^8x_4^{128}, x_4^{512}, x_1x_3^{32}x_5^{512}, x_2x_4^{128}x_5^{512}, x_3^{32}x_4^{128}x_5^{512}, \\ x_5^{2048}, x_2^8x_3^{32}x_6^{2048}, x_1x_4^{128}x_6^{2048}, x_3^{32}x_4^{128}x_6^{2048}, x_1x_5^{512}x_6^{2048}, x_2^8x_5^{512}x_6^{2048}).$$

Then the Betti numbers $\beta_{2,2729}^{S \otimes_{\mathbb{Z}} \ell}(J(S \otimes_{\mathbb{Z}} \ell))$ and $\beta_{3,2729}^{S \otimes_{\mathbb{Z}} \ell}(J(S \otimes_{\mathbb{Z}} \ell))$ (which correspond to $\beta_{2,6}^{S \otimes_{\mathbb{Z}} \ell}(I(S \otimes_{\mathbb{Z}} \ell))$ and $\beta_{3,6}^{S \otimes_{\mathbb{Z}} \ell}(I(S \otimes_{\mathbb{Z}} \ell))$, respectively) are nonzero precisely when $\text{char } \ell = 2$; otherwise they are zero. \square

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